ModSim Thermal Systems Case Study

Situation: We want to fry a huge donut in oil. It has been suggested that a dumpster filled partially with oil heated from below will work.

Questions: How long will it take to heat the oil to the frying temperature, and how does this depend on the amount of heat delivered over time? Once the oil is hot, and the donut is added, how long will it take to cook the donut?

Figure/System Diagram:
Differential Equation:

\[
\frac{dU}{dt} = \dot{Q}_{\text{in}} - \dot{Q}_{\text{surf}} - \dot{Q}_{\text{dump}}
\]

Think about each term, and what parameters we need to set, and the assumptions we'll need to make:

\dot{Q}_{\text{in}} \quad [\text{W}]

this can be set as some reasonable value to start, and we could run our model for a range of \dot{Q}_{\text{in}} to see how it affects the time needed to reach fry temperature.

\dot{Q}_{\text{surf}} \quad [\text{W}]

Convection from surface of oil to air
Use Newton's Law of Cooling to estimate this.

\[
\dot{Q}_{\text{surf}} = h A_s (T_{\text{surf}} - T_{\infty})
\]

\(A_s = \text{surface area of oil exposed to air, we'll obviously need to know size/shape of chippster}\)

\(T_{\text{surf}} = \text{temperature of oil. In fact, this is just } T_{\text{oil}}(t) \text{ since we will be making the lumped assumption. We won't set this and we'll solve for it}\)

\(T_{\infty} = \text{temperature of the surroundings, we will just choose a reasonable parameter for this}\)

\(h = \text{heat transfer coefficient that describes how much heat is lost from a flat surface to still air. We'll choose a reasonable value for this to start}\)
Heat lost through dumpster wall

This is a little more complicated, so let's sketch what is happening.

The heat is transferred through the dumpster wall and is then carried away via convection from the outer dumpster wall. The temperature profile from $T_{oil}$ to $T_{w,0}$ to $T_{oo}$ indicates this.

The easiest way to handle this situation is by using the idea of thermal resistance or thermal circuits. Take the simple example below.

Using Fourier's Law, we could determine that: $\dot{Q} = \frac{KA}{l} \Delta T$

where $A$ is the plane area of the wall.
\[
\dot{Q}_{\text{steel}} = \frac{k_{\text{s}}A}{\ell} \Delta T \\
\dot{Q}_{\text{foam}} = \frac{k_{\text{f}}A}{\ell} \Delta T
\]

The only difference is the thermal conductivity, \( k \), which tells us how heat passes through a \( 1 \) m wall with a \( 1 \) K temperature difference:

\[
k = \left[ \frac{W}{mK} \right]
\]

\[
k_{\text{steel}} \approx 40 \frac{W}{mK} \\
k_{\text{foam}} \approx 0.04 \frac{W}{mK}
\]

So which material will “allow” more heat to transfer through it?

**Ans:** The steel, of course!

We can define a thermal resistance, similar to an electrical resistance:

\[
V = IR \implies \Delta T = \frac{\dot{Q}}{R}
\]

or \( \dot{Q} = \frac{\Delta T}{R} \)

In the case of conduction through a wall, \( R = \frac{\ell}{kA} \)

Like resistances in series, the effect of the composite wall is additive.

\[
R_{\text{tot}} = R_A + R_B = \frac{\ell_A}{k_{\text{AA}}} + \frac{\ell_B}{k_{\text{BB}}} \quad \text{AND} \quad \dot{Q} = \frac{\Delta T_{\text{tot}}}{R_{\text{tot}}} (T_c - T_h)
\]
What about convection? It is also like a resistor in series:

\[
\dot{Q}_{\text{cond}} = -\frac{KA}{l} (T_{w,0} - T_n) \quad \Rightarrow \quad R_{\text{cond}} = \frac{l}{KA}
\]

\[
\dot{Q}_{\text{conv}} = hA (T_{w,0} - T_\infty) \quad \Rightarrow \quad R_{\text{conv}} = \frac{1}{hA}
\]

So \(\dot{Q}_{\text{cond}} = \dot{Q}_{\text{conv}} = \frac{Q}{R_{\text{tot}}} = \frac{\Delta T_{\text{tot}}}{R_{\text{tot}}} = \frac{T_\infty - T_n}{\frac{l}{KA} + \frac{1}{hA}}\)

Going back to \(\dot{Q}_{\text{dump}}\), we can apply this equation.

\(T_n\) = temperature of oil \(T_{oil}(t)\) (solve for)

\(T_\infty\) = temperature of surrounding air (set)

\(l\) = wall thickness of dumpster (set)

\(K\) = thermal conductivity of dumpster (look up)

\(A\) = area of dumpster wall that heat is lost through (set)

\(h\) = heat transfer coefficient for convection from a hot wall to still air (estimate)
Finally, we need a way to connect our stock (internal energy of the oil) to the temperature of the oil.

\[
\Delta U_{\text{oil}} = M_{\text{oil}} C_{\text{oil}} \Delta T_{\text{oil}}
\]

\[
U(t) - U(0) = M_{\text{oil}} C_{\text{oil}} (T_{\text{oil}}(t) - T_{\text{oil}}(0))
\]

This term will always just be referred to as \( \Delta U_{\text{oil}} \), with the understanding that it is a delta from the internal energy at time zero.

\( M_{\text{oil}} \) = mass of oil

\( C_{\text{oil}} \) = specific heat of oil

Putting it all together:

\[
\frac{d(\Delta U)}{dt} = \dot{Q}_{\text{in}} - \dot{Q}_{\text{surf}} - \dot{Q}_{\text{dump}}
\]

\[
\frac{d(\Delta U)}{dt} = \dot{Q}_{\text{in}} - h_{\text{surf}} A_{\text{surf}} (T_{\text{oil}}(t) - T_{\infty})
\]

\[
\frac{1}{K_{\text{dump}} A_{\text{dump}}} + \frac{1}{h_{\text{dump}} A_{\text{dump}}}
\]

I have a choice here. I can substitute in for \( T_{\text{oil}}(t) \) so the entire equation is in terms of \( \Delta U \), or I can substitute in for \( \Delta U \) so the entire equation is in terms of \( T_{\text{oil}}(t) \). Let's do the latter, since \( T_{\text{oil}} \) is more important to us as a result.
(Although it is just as easy to convert AU to Toil after the results have been produced.)

\[
\frac{d}{dt} \left[ M_{oil} \, C_{oil} \left( T_{oil}(t) - T_{oil}(0) \right) \right] = Q_{in} - h_{s} A_{s} (T_{oil}(t) - T_{\infty}) - \frac{L}{K_{oil} A_{oil} + h_{oil} A_{oil}}
\]

the second term, \( m_{oil} C_{oil} T_{oil}(0) \) is a constant, and the derivative of a constant is zero, so this simplifies to:

\[
M_{oil} \, C_{oil} \frac{dT_{oil}}{dt} = Q_{in} - h_{s} A_{s} (T_{oil}(t) - T_{\infty}) - \frac{L}{K_{oil} A_{oil} + h_{oil} A_{oil}}
\]

STOP!

What assumptions did we have to make to get here?

1. Oil is a lumped system with a uniform temperature throughout.

2. There is no wind; we've assumed still air.

3. We are currently ignoring the thermal mass of the dumpster and how long it might take to heat up the metal.

4. All heat from the burner (or whatever the heat source is) goes into heating the oil; none is lost to other means. (i.e. heating the ground under the burner)

5. Ignoring heat lost by radiation from hot oil and hot dumpster.
CHOOSING REASONABLE PHYSICAL PARAMETERS

Temperatures:

\[ T_\infty = 70^\circ F \]
\[ T_{\text{frying}} = 375^\circ F \] (temp. needed to fry donut)
\[ T_{\text{oil}(0)} = 70^\circ F \] (initial temp. of oil)

Geometry:

Dumpster: \( 7' \times 7' \times 4' \), half filled with canola oil
width length height
wall thickness = 2 "

Material Properties:

Canola oil: \[ \rho_{\text{oil}} = 0.92 \text{ g/cm}^3 \] (density)
\[ C_{\text{oil}} = 1.9 \text{ KJ/kg} \] (specific heat)

Steel dumpster: \[ K_d \approx 40 \text{ W/m K} \] (thermal conductivity)

Heat Transfer Parameters:

\[ h_d \approx 10 \text{ W/m}^2 \text{ K} \] heat transfer coefficients for convection
\[ h_s \approx 10 \text{ W/m}^2 \text{ K} \]

\[ \dot{Q}_{\text{in}} \approx 50,000 \text{ Btu/h}^2 \] starting guess, will likely run model for a range of \( \dot{Q}_{\text{in}} \)
typical heat put out by a gas grill
I. Rearrange differential so I can apply Forward Euler:

$$\frac{dT_{oil}}{dt} = \dot{Q}_{in} \frac{m_{o_c}}{m_{o_c}} - (T_{oil} - T_{\infty}) \left[ h_S A_s + \frac{1}{k_d A_d} + \frac{1}{h_d A_d} \right]$$

II. Forward Euler

$$T_{n+1} = T_n + \Delta t \quad \text{(time steps, will need to choose appropriate } \Delta t)$$

$$T_{n+1} = T_n + \left[ \dot{Q}_{in} \frac{m_{o_c}}{m_{o_c}} - (T_n - T_{\infty}) \left[ h_S A_s + \frac{1}{k_d A_d} + \frac{1}{h_d A_d} \right] \right] \Delta t$$

III. Check units and calculate needed parameters

$$\dot{Q}_{in} = 15000 \text{ W} \quad (\approx 50,000 \text{ Btu/hr})$$

$$m_o = \rho_o V_o = \left( 0.92 \frac{g}{cm^3} \right) \left( 7 \text{ ft} \right) \left( 7 \text{ ft} \right) \left( 2 \text{ ft} \right) \left( \frac{30.48 \text{ cm}}{1 \text{ ft}} \right)^3$$

$$m_o = 2553040 \text{ g} = 2553 \text{ kg}$$

$$T_0 = 70^\circ F \approx 295 \text{ K}$$

$$T_f = 375^\circ F \approx 465 \text{ K}$$

$$T_{\infty} = 70^\circ F \approx 295 \text{ K}$$

$$l = 2 \text{ m} = 0.0508 \text{ m}$$

$$A_s = \left( 7 \text{ ft} \right) \left( 7 \text{ ft} \right) \left( \frac{30.48 \text{ cm}}{1 \text{ ft}} \right)^2 = 622 \text{ cm}^2 = 4.55 \text{ m}^2$$

$$A_s = 4 \left( 7 \text{ ft} \right) \left( 2 \text{ ft} \right) \left( \frac{30.48 \text{ cm}}{1 \text{ ft}} \right)^2 = 5.20 \text{ m}^2$$

IV. MATLAB TIME!!